

Scaling properties in dynamics of non-analytic complex maps near the accumulation point of the period-tripling cascade

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Abstract

The accumulation point of the period-tripling bifurcation cascade in complex quadratic map was discovered by Golberg, Sinai, and Khanin (Russ.Math.Surv. 38:1, 1983, 187), and independently by Cvitanović and Myrheim (Phys.Lett. A94:8, 1983, 329). As we argue, in the extended parameter space of smooth maps, not necessary satisfying the Cauchy – Riemann equations, the scaling properties associated with the period-tripling are governed by two relevant universal complex constants. The first one is $\delta_1 \cong 4.6002 - 8.9812i$ (in accordance with the mentioned works), while the other one, responsible for the violation of the analyticity, is found to be $\delta_2 \cong 2.5872 + 1.8067i$. It means that in the extended parameter space the critical behaviour associated with the period-tripling cascade is a phenomenon of codimension four. Scaling properties of the parameter space are illustrated by diagrams in special local coordinates. We emphasize a necessity for a nonlinear parameter change to observe the parameter space scaling.

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1 Introduction

One of the most popular illustrations in the nonlinear science is a picture of the Mandelbrot set [1, 2] (Fig. 1). On the complex parameter plane it is defined as the set of points λ , at which the iterations of the complex quadratic map

$$z' = \lambda - z^2, \quad \lambda, z \in \mathbb{C} \quad (1)$$

launched from the origin $z = 0$ never diverge. (Here the prime marks the dynamical variable relating to the next iteration, i.e. to the next moment of discrete time.)

The Mandelbrot set has subtle and complicated structure, which is a subject of extensive researches. Objects analogous to the Mandelbrot set are present in parameter planes of other complex analytic iterative maps too [2, 3].

The Mandelbrot set is regarded as a classic example of fractal; this suggests a sort of self-similarity, or scaling.¹ One particular manifestation of scaling is the Feigenbaum period-doubling cascade. The leaves of the Mandelbrot cactus placed along the real axis correspond to periods 2, 4, 8,... 2^k ..., and, in asymptotics of large k , they perfectly reproduce each other under subsequent magnification by a real universal factor $\delta_F \cong 4.6692$. The limit point of the period-doubling accumulation is disposed on the real axis at $\lambda_F = 1.401155...$, and it is called the Feigenbaum critical point. The scaling properties follow from the renormalization group (RG) analysis developed by Feigenbaum [6, 7]. Behaviour of the solution of the RG equation in complex domain was discussed e.g. by Nauenberg [8] and by Wells and Overill [9].

One can find many other critical points in the parameter plane of the map (1), at which the Mandelbrot cactus displays other scaling regularities. In particular, we can select a path on the complex plane across the sequence of leaves corresponding to periods 3, 9, 27..., 3^k ..., and arrive at the period-tripling accumulation point

$$\lambda = 0.0236411685377 + 0.7836606508052i. \quad (2)$$

¹In fact, as shown by Milnor [4], the subtle structure ("hairiness") of the Mandelbrot set does not reproduce itself on deep levels of resolution in small-scales. Speaking more accurately, one should relate the property of self-similarity rather to a definite subset called the Mandelbrot cactus [5]. It consists of a domain of existence of a stable fixed point and of domains associated with attractive periodic orbits, which originate from the fixed point via all possible bifurcation sequences (see the grey colored part of the picture in Fig. 1).

As this point was first discovered by Golberg, Sinai, and Khanin [10], we call it the *GSK critical point*. Independently, it was found and studied also by Cvitanović and Myrheim [5, 11]. As in the case of period-doubling, the RG analysis has been applied, and self-similarity of the sequence of leaves of the Mandelbrot cactus has been revealed. According to Refs. [10, 11, 5], the scaling factor in the case of period-tripling is complex, $\delta_1 \cong 4.6002 - 8.9812i$.²

Sometimes it may be preferable to deal not with the one-dimensional complex analytic maps, but with the equivalent class of two-dimensional real maps satisfying the Cauchy – Riemann equations.

It would be interesting to discuss a possibility of observation of phenomena associated with complex analytic dynamics and the Mandelbrot set in physical systems. Recently this question was posed by Beck [12]. He has considered motion of a particle in a double-well potential in a time-depended magnetic field. As he has stated, under certain assumptions dynamics of the particle may be described by the complex quadratic map or by other complex analytic maps. It should be emphasized, however, that discussing physical applications for any special type of dynamics we should take into account a robustness of a phenomenon under study. In this context it is important to know either phenomena associated with the complex analytic dynamics survive or not in slightly modified maps, which can violate the Cauchy – Riemann conditions.

In the original work [10] the authors claimed that the infinite period-tripling cascade could occur typically in two-parameter families of real two-dimensional maps (see also [13]). In contrast, from studies of Peinke et al. [14], Klein [15], and from qualitative analysis of Cvitanović and Myrheim [5] it follows that in non-analytic maps the parameter space arrangement is changed drastically, and, apparently, there is no opportunity of surviving for the universality intrinsic to the period-tripling. Recently Peckham and Montaldi have presented extensive bifurcation analysis of a complex quadratic map with a non-analytic term [16, 17]. However, the question what happens with the bifurcation cascades of period-tripling remains not clear.³ Sure enough, this matter deserves accurate quantitative analysis in terms of the RG approach, and

²Beside (2), a complex conjugate period-tripling accumulation point exists. It is placed at $\lambda = \lambda_c^*$, and scaling in its neighborhood is governed by the conjugate constant δ_1^* .

³The same assertion relates to other multi-tupling bifurcation cascades, although we do not touch them in the present paper.

this is the main goal of the present article. Our final conclusion is that in real two-dimensional maps the critical behaviour associated with the infinite period-tripling cascade will occur as a phenomenon of codimension 4, i.e. it may be found at some special points of a four-dimensional parameter space. In other words, the codimension is so high that it seems not easy to observe the phenomenon in physical systems, at least in absence of some special symmetries.

In section 2 we recall contents of the RG analysis developed in [10, 11, 5]. Then, it is extended to study the fixed point solution of the RG equation in respect to a class of perturbations, which can violate the Cauchy – Riemann conditions. We find that among the non-analytic eigenmodes of the linearized RG transformation one is of relevance. In section 3 we formulate the model map appropriate for a study of the extended parameter space. Then we explain a necessity of nonlinear variable change to define local coordinates for observation of scaling in the parameter space, and construct a procedure to find this coordinate change numerically. In section 4 we discuss some details of the parameter space arrangement near the GSK critical point and present graphical illustrations of the intrinsic scaling properties.

2 Renormalization group analysis

To start, let us follow Refs. [10, 11, 5] and assume that discrete time evolution of the complex variable z is governed by the complex quadratic map (1). Then, for three iterations we have

$$z' = f(f(f(z))) = \lambda - (\lambda - (\lambda - z^2)^2)^2. \quad (3)$$

Next, we rescale the variable z by a factor α_0 and choose this factor to normalize the new mapping to unity at the origin. Then we have

$$z' = f_1(z), \quad (4)$$

where

$$f_1(z) = \alpha_0 f(f(f(z/\alpha_0))), \quad \text{and} \quad \alpha_0 = 1/f(f(f(0))). \quad (5)$$

This is just one step of the RG transformation.

Now we can apply the same procedure to the function $f_1(z)$ and obtain the next function $f_2(z)$, and so on. By multiple repetition of the transformation we arrive at the recurrent

functional equation

$$f_{k+1}(z) = \alpha_k f_k(f_k(f_k(z/\alpha_k))), \quad \alpha_k = 1/f_k(f_k(f_k(0))). \quad (6)$$

Here $f_k(z)$ represents the evolution operator for 3^k iterations of the original map in terms of the renormalized dynamical variable. Note that the rescaling constants α_k , which appear at the subsequent steps of the procedure, are complex.

According to Refs. [10, 11, 5], at the GSK critical point the sequence $f_k(z)$ converges to the fixed point function $g(z) = \lim_{k \rightarrow \infty} f_k(z)$, which obeys the equation

$$g(z) = \alpha g(g(g(z/\alpha))), \quad \alpha = \lim_{k \rightarrow \infty} \alpha_k = 1/g(g(g(0))). \quad (7)$$

It is a generalization of the Feigenbaum-Cvitanović equation [6, 7] for the case of period-tripling. This functional equation may be solved numerically by means of approximating the function $g(z)$ via a finite polynomial expansion and computing the coefficients with a use of multi-dimensional Newton method [10, 11]. We have reproduced these calculations, and present the results in Table I. Note that only terms of even power are nonzero.

As follows from the computations, the scaling constant is

$$\alpha = -2.09691989 + 2.35827964i \quad (8)$$

in good agreement with Refs. [10, 11, 5].

As the rescaling constant is known, it is convenient to redefine the RG transformation. Now we will assume that the rescaling factor is the same at all subsequent steps of the procedure and equals to the universal value (8). We just substitute $\alpha_k = \alpha$ into Eq.(6) and obtain

$$f_{k+1}(z) = \alpha f_k(f_k(f_k(z/\alpha))). \quad (9)$$

Obviously, this version of the RG transformation possesses the same fixed point $g(z)$, although the normalization $g(0) = 1$ should be regarded now as an arbitrarily accepted additional condition.

Let us consider the map $z' = g(z)$ corresponding to the fixed-point solution of the RG equation. It may be checked numerically that this map has an unstable fixed point $z = z_* \cong 0.691473 - 0.302692i$, and derivative at this point is

$$\mu_c = g'(z_*) = -0.47653180 - 1.05480868i. \quad (10)$$

Then, as follows from Eq. (7), z_*/α is a point belonging to an orbit of period 3. The Floquet eigenvalue (or multiplier) for this cycle will be the same complex number μ_c . By induction, it is easy to prove that there is an infinite countable set of the period- 3^k cycles (z_*/α^k belongs to the period- 3^k orbit), with the same value of the multiplier.

As the mapping $z' = g(z)$ represents the asymptotic form of the evolution operators for the original map (1), it must have an infinite set of unstable orbits of periods 3^k at the GSK critical point. The same is true for any other map relating to the same universality class. The asymptotic value of the multipliers for the set of cycles is the universal constant μ_c .

Now let us assume that we have a real smooth two-dimensional map

$$x' = \varphi(x, y), \quad y' = \psi(x, y), \quad (11)$$

which depends on some parameters. In terms of the complex variable this map may be expressed, in general, via a function of two arguments $z = x + iy$ and $z^* = x - iy$: $z' = F(z, z^*)$, where $F(x + iy, x - iy) = \varphi(x, y) + i\psi(x, y)$. Next, we suppose that at some special values of parameters the map becomes complex analytic, i.e. the Cauchy – Riemann equations hold: $\varphi_x = \psi_y$, $\varphi_y = -\psi_x$, and the dynamics is just of the kind intrinsic to the GSK critical point. It means that for a number of iterations 3^k (k large) the evolution operator is represented in appropriate normalization by the universal function $g(z)$. Now, if we slightly change the values of parameters and depart from the original point in the parameter space, some perturbation to the fixed point of the RG transformation will appear. In general, the perturbation may not satisfy the Cauchy – Riemann conditions, so the evolution operator for 3^k steps of iterations must be written as

$$F_k(z) = g(z) + \epsilon h_k(z, z^*). \quad (12)$$

Here $\epsilon \ll 1$, and h_k is a smooth function of two arguments. By three-fold iteration of the map (12) with rescaling of z by factor α we arrive (in the first order in ϵ) to the same form of the operator, but with the modified perturbation term:

$$\begin{aligned} h_{k+1}(z, z^*) = & \alpha[g'(g(g(z/\alpha)))g'(g(z/\alpha))h_k(z/\alpha, (z/\alpha)^*) + \\ & + g'(g(g(z/\alpha)))h_k(g(z/\alpha), (g(z/\alpha))^*) + h_k(g(g(z/\alpha)), (g(g(z/\alpha)))^*)], \end{aligned} \quad (13)$$

where g' means the derivative of the function g . Obviously, this relation has a structure

$$h_{k+1} = \hat{m}h_k \quad (14)$$

where \hat{m} is a linear operator defined by the right-hand part of (13).

The behaviour of h_k under subsequent iterations of the linear operator is linked with the spectrum of eigenvalues. Indeed, let us have an eigenfunction $h(z, z^*)$ associated with an eigenvalue ν :

$$\begin{aligned} \nu h(z, z^*) = & \alpha [g'(g(z/\alpha))g'(g(z/\alpha))h(z/\alpha, (z/\alpha)^*) + \\ & + g'(g(z/\alpha))h(g(z/\alpha), (g(z/\alpha))^*) + h(g(z/\alpha), (g(z/\alpha))^*)^*)]. \end{aligned} \quad (15)$$

Then, the functional sequence h_k may contain a component, which behaves as $\nu^k h(z, z^*)$. Those eigenvalues will be of relevance, which are larger than unity in modulus, and which are not associated with infinitesimal variable changes.

To solve the eigenproblem numerically we may expand $h(z, z^*)$ over powers of z and z^* , and use the polynomial representation for $g(z)$, which is already known. As we keep a finite number of terms in the expansions, the eigenproblem reduces to a finite-dimensional one. Then, the spectrum of eigenvalues may be calculated by means of standard methods of linear algebra.

As the computations show, the eigenfunction associated with the senior eigenvalue does not depend on the second argument z^* . The coefficients of the power expansion are listed in Table 2; only even-power terms are nonzero. It is clear that this eigenfunction represents a perturbation, which retains the map in the class of complex-analytic functions. The computed eigenvalue is

$$\delta_1 = 4.60022558 - 8.98122473i, \quad (16)$$

in good coincidence with the data of Refs. [10, 11, 5].

It appears, however, that one more relevant eigenfunction exists with eigenvalue larger than unity in modulus, and the polynomial expansion contains powers of both arguments z and z^* (Table 3). This solution is responsible for the non-analytic perturbation to the fixed point of the RG equation. As we shall see, the eigenvalue may be expressed explicitly via the scaling constant α .

We know that the fixed-point function $g(z)$ is even. Hence, it follows from (15) that

$$\begin{aligned} \nu h(-z, -z^*) = & \alpha [g'(g(z/\alpha))g'(g(z/\alpha))h(-z/\alpha, -(z/\alpha)^*) + \\ & + g'(g(z/\alpha))h(-g(z/\alpha), -(g(z/\alpha))^*) + h(-g(z/\alpha), -(g(z/\alpha))^*)^*)]. \end{aligned} \quad (17)$$

We may represent the eigenfunction as

$$h(z, z^*) = h^s(z, z^*) + h^a(z, z^*), \quad (18)$$

where

$$h^s(z, z^*) = \frac{h(z, z^*) + h(-z, -z^*)}{2}, \quad (19)$$

is a symmetric part, and

$$h^a(z, z^*) = \frac{h(z, z^*) - h(-z, -z^*)}{2}. \quad (20)$$

is an antisymmetric component. Then, subtracting (17) from (15) we have

$$\nu h^a(z, z^*) = \alpha g'(g(z/\alpha))g'(g(z/\alpha))h^a(z/\alpha, (z/\alpha)^*). \quad (21)$$

Now let us set

$$h^a(z, z^*) = \frac{g'(z)}{z} \Phi(z, z^*). \quad (22)$$

We have $g'(z) \propto z$, so the ratio $g'(z)/z$ does not possess a singularity at the origin. Using the relation

$$g'(g(z/\alpha))g'(g(z/\alpha))g'(z/\alpha) = 1, \quad (23)$$

which follows from (7), we obtain a simple equation for $\Phi(z, z^*)$:

$$\nu \Phi(z, z^*) = \alpha^2 \Phi((z/\alpha), (z/\alpha)^*). \quad (24)$$

Obviously, any product $z^M(z^*)^N$ with integer $M \geq 0, N > 0, M + N$ odd, represents an eigenfunction, and the associated eigenvalue is $\alpha^{2-M}(\alpha^*)^{-N}$. Observe that one of these numbers is of modulus larger than 1. Indeed, setting $M = 0$ and $N = 1$, i.e. $\Phi = z^*$, we have

$$h^a(z, z^*) = \frac{g'(z)}{z} z^*, \quad (25)$$

and

$$\nu = \frac{\alpha^2}{\alpha^*} = \delta_2 = 2.58728651 + 1.80679396i. \quad (26)$$

The antisymmetric part of the eigenfunction, as well as the eigenvalue δ_2 , coincide with those obtained numerically (Table 3).

All eigenvalues beside δ_1 and δ_2 are irrelevant. Some of them are associated with infinitesimal variable changes (cf. [7]). They may be expressed explicitly via the fixed point function $g(z)$, see Table 4. Numerical calculations show that all other eigenvalues are of moduli less than 1.

The condition for realization of the critical behaviour in some map is a possibility of tuning its parameters to make complex coefficients at two relevant eigenvectors associated with δ_1 and

δ_2 equal zero. It yields two complex, or four real, equations on the parameters. Generically, the solution will exist if the number of unknowns is not less than the number of equations. So, we conclude that the phenomenon may occur as a generic in a space of four real parameters. In other words, it is of codimension four.

In our line of reasoning we have started from the analytic map. Nevertheless, analyticity is not a necessary condition for the GSK critical behaviour. To clarify this remark, let us take the analytic map $g(z)$ (the fixed point function of the RG equation) and introduce a non-analytic perturbation, a small term represented by a smooth function of two arguments, z and z^* . It may be chosen in such a special way that there will be no contribution from the growing non-analytic eigenmode associated with the eigenvalue δ_2 . (See Eq.(24) and the following discussion.) Then, under subsequent iterations of the RG transformation the perturbation evolving in accordance with Eq.(13) will decay. It means that the evolution operator will tend to the same fixed point function as in the analytic map.

Let us consider a concrete example, the map studied by Gunaratne [13]

$$\begin{aligned} x' &= a + x^2 - y^2, \\ y' &= b + (2 + \epsilon)xy. \end{aligned} \tag{27}$$

By means of variable changes $z = x + yi$, $\lambda = a + ib$ and $\varepsilon = \epsilon/4$ this map can be rewritten as $z' = \lambda + z^2 + \varepsilon(z^2 - (z^*)^2)$. Observe that the non-analytic term $(z^*)^2$ is quadratic, not linear, hence, it can not give rise to the eigenvector with eigenvalue δ_2 . As to the contribution into the growing analytical eigenmode (eigenvalue δ_1), it may be compensated by an appropriate shift of $\lambda = a + ib$. Thus, the GSK point must exist; this assertion is in accordance with the conclusion of Ref.[13]. As we have found, for $\epsilon = 0.2$ the critical point is located at $a \cong -0.0355$, $b \cong 0.7477$.

3 Model map and local scaling coordinates at the GSK critical point

Now we will construct a model map to study a vicinity of the GSK critical point in the extended parameter space.

If we consider a shift of λ from the critical value in the complex quadratic map (1), then only a perturbation given by the eigenfunction $h_1(z)$ can arise. As we wish the perturbation $h_2(z, z^*)$ to appear, it is necessary to add an appropriate non-analytic term. This eigenfunction behaves at small $|z|$ as $h_2 \propto z^*$, hence, we reach this goal adding the term proportional to z^* . Thus, we come to the model map

$$z_{n+1} = \lambda - z_n^2 + \varepsilon z_n^*, \quad (28)$$

which may be regarded as unfolding of the map (1). The extended parameter space is the two-dimensional complex space $\mathbb{C}^2 : (\lambda, \varepsilon)$. Up to a trivial variable change ($z \rightarrow -z$) this is the same map that the authors of Refs. [16, 17] have chosen for their extensive bifurcation analysis. It is equivalent to a four-parameter two-dimensional real map

$$x_{n+1} = a - x_n^2 + y_n^2 + ux_n + vy_n, \quad y_{n+1} = b - 2x_n y_n + vx_n - uy_n, \quad (29)$$

where we set $z = x + iy$, $\lambda = a + ib$, $\varepsilon = u + iv$.

As we have two relevant eigenfunctions $h_1(z)$ and $h_2(z, z^*)$, the sequence of the evolution operators generated by iterations of the RG transformation will behave as

$$f_k(z, z^*) = g(z) + C_1(\lambda, \varepsilon) \delta_1^k h^{(1)}(z) + C_2(\lambda, \varepsilon) \delta_2^k h^{(2)}(z, z^*); \quad (30)$$

this is true in linear approximation in respect to the deflection from the fixed point solution $g(z)$. Here C_1 and C_2 are some complex coefficients. They vanish at the GSK critical point ($\lambda = \lambda_c, \varepsilon = 0$), but become nonzero as we depart from it in the parameter space.

It would be convenient to use C_1 and C_2 as local coordinates in a neighborhood of the critical point. In these coordinates the scaling properties of the parameter space are very evident. Indeed, let us change C_1 to C_1/δ_1 and C_2 to C_2/δ_2 . Then, according to (30) the evolution operator f_{k+1} will accept the same form as the operator f_k at the old point. It means that the similar dynamical regimes will occur at the new point, but with tripled characteristic time scale.

Unfortunately, we do not know explicit expressions of the coefficients C_1 and C_2 via the parameters of the model map (28). Hence, the coordinate system suitable for demonstration of scaling (*the scaling coordinates*) must be found numerically with a sufficient precision.

Let us consider a manifold M in the parameter space determined by a condition $C_1(\lambda, \varepsilon) = 0$, or $C_1(a, b; u, v) = 0$, where we set $\lambda = a + ib$ and $\varepsilon = u + iv$. We can rewrite this equation as

$$a = a_c + F(u, v), \quad b = b_c + G(u, v), \quad (31)$$

where a_c and b_c correspond to the GSK critical point.

In fact, we may cut the Taylor expansions for $F(u, v)$ and $G(u, v)$ keeping the terms of the first and second order. The reason is the following. As we stay at the manifold M , the renormalization of the complex coordinate $\varepsilon = u + iv$ by factor δ_2 has to ensure realization of similar dynamical regimes. Let us suppose that we miss some term of power m in the expansion of F or G . When we perform the scale change $\varepsilon \rightarrow \varepsilon/\delta_2^k$, this term will be of order δ_2^{-mk} . So, the error in the amplitude of the senior eigenvector will behave as δ_1^k/δ_2^{mk} . In the case of our concrete type of criticality (GSK) it is dangerous only for $m \leq 2$, otherwise the error asymptotically vanishes as $k \rightarrow \infty$. Indeed, in accordance with (16) and (26) we have $|\delta_2| < |\delta_1|$ and $|\delta_2|^2 < |\delta_1|$, but $|\delta_2|^3 > |\delta_1|$. So, only terms of the first and second order in the Taylor expansion must be evaluated accurately,⁴ and we may write

$$\begin{aligned} F(u, v) &= Au + Bv + Pu^2 + Quv + Rv^2, \\ G(u, v) &= Cu + Dv + Su^2 + Tuv + Uv^2. \end{aligned} \quad (32)$$

Here the capital letters A, B, \dots, U designate some constants, which may be found numerically.

As mentioned, precisely at the GSK critical point the map has an infinite sequence of unstable periodic orbits. Let us come out this point and look at some cycle of period $N = 3^k$. In terms of real perturbation vector $(\tilde{x}, \tilde{y}) = (\text{Re } \tilde{z}, \text{Im } \tilde{z})$ the evolution over one period is determined by the Jacobi matrix

$$\mathbf{J} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \partial x_N / \partial x_0 & \partial x_N / \partial y_0 \\ \partial y_N / \partial x_0 & \partial y_N / \partial y_0 \end{pmatrix}. \quad (33)$$

While the map is complex analytic, the Cauchy – Riemann equations hold, and $a_{11} = a_{22}$, $a_{12} = -a_{21}$. Otherwise, in presence of the term εz^* , these equalities are violated.

Let us consider trace

$$S = a_{11} + a_{22} \quad (34)$$

⁴See other examples of nonlinear parameter change for observation of the multi-parameter scaling in Refs. [22, 23, 24].

and determinant

$$J = a_{11}a_{22} - a_{12}a_{21} \quad (35)$$

of the matrix \mathbf{J} at the cycle. Surely, these values depend on the parameters of the map: $S = S(\lambda, \varepsilon) = S(a, b, u, v)$ and $J = J(\lambda, \varepsilon) = J(a, b, u, v)$. Precisely at the critical point ($a = a_c, h = b_c, u = 0, v = 0$) they tend, respectively, to $2\text{Re } \mu_c$ and $|\mu_c^2|$, as k grows.

Let us suppose that we know the first and the second derivatives of the trace and determinant in respect to the parameters $u = \text{Re } \varepsilon$ and $v = \text{Im } \varepsilon$ for two sufficiently large periods, 3^{k+1} and 3^k . We require the following scaling property to hold. If one takes (i) some infinitesimal value of $\varepsilon = \varepsilon_0 = u + iv$ and (ii) the rescaled value $\varepsilon = \delta_2 \varepsilon_0 = (\delta'_2 u - \delta''_2 v) + i(\delta'_2 v + \delta''_2 u)$, where $\delta'_2 = \text{Re } \delta_2$, $\delta''_2 = \text{Im } \delta_2$, then traces and determinants for both cycles must coincide at arbitrarily chosen u and v :

$$\begin{aligned} S^{k+1}(u, v) &= S^k(\delta'_2 u - \delta''_2 v, \delta'_2 v + \delta''_2 u), \\ J^{k+1}(u, v) &= J^k(\delta'_2 u - \delta''_2 v, \delta'_2 v + \delta''_2 u). \end{aligned} \quad (36)$$

Now we substitute the Taylor expansion $S(u, v) = S_0 + S_u u + S_v v + \frac{1}{2} S_{uu} u^2 + S_{uv} uv + \frac{1}{2} S_{vv} v^2$, and analogous representation for $J(u, v)$, and equalize separately coefficients at the terms of the first and of the second power in u and v . It yields ten real equations with ten unknowns A, B, \dots, U . We have found numerically the derivatives $(S_u, S_v, S_{uu}, \dots, J_u, J_v, J_{uu}, \dots)$ for sufficiently large periods and solved the mentioned equations. The resulting values of the constants were found to be asymptotically stable, estimated as

$$\begin{aligned} A &= -0.232750391, B = 0.02699484, C = -B, D = A, \\ P &= -0.2173, Q = -0.070, R = -0.3184, \\ S &= 0.1533, T = 0.1010, U = 0.0830. \end{aligned} \quad (37)$$

(In our computations the maximal periods we could handle were about 3^8 or 3^9 . However, the convergence is rather fast, and already at $3^4 \div 3^7$ we obtain sufficiently accurate data.)

Then, the the model map is represented as

$$\begin{aligned} x' &= a - x^2 + y^2 + ux + vy + Au + Bv + Pu^2 + Quv + Rv^2, \\ y' &= b - 2xy + vx - uy + Cu + Dv + Su^2 + Tuv + Uv^2, \end{aligned} \quad (38)$$

and the coordinate system (a, b, u, v) is appropriate for the demonstration of scaling. Indeed, a shift of u and v will contribute exceptionally into the second eigenvector, and a shift of a and b — only into the first eigenvector.

4 Scaling properties of the extended parameter space in a neighborhood of the critical point

The number of real parameters in the model map (38) is four, and complete bifurcation analysis of the parameter space (a, b, u, v) would be complicated and tedious (see, nevertheless, Refs. [16, 17]). Here we only want to present computer illustrations for scaling properties following from the results of Sec.2. We will consider some two-dimensional cross-sections of the parameter space and look at the "topography charts" for dynamical regimes on these surfaces.

For graphical presentation we will use the technique of Lyapunov charts (see [19, 20, 21] for previous applications of this method). As we have chosen the parameter-space cross-section to be studied, we compute the senior Lyapunov exponent for the model map at each pixel of the two-dimensional plot, and mark this pixel by an appropriate grey tone. We code the Lyapunov exponent from large negative values to zero by tones from dark to light grey (it corresponds to periodic regimes). White color represents zero Lyapunov exponent (e.g. quasiperiodicity or states at the onset of chaos), and black — positive values (chaos). Such a convention ensures a clear vision of the border between regular and chaotic dynamics. In some domains of the parameter space the model map manifests divergence; these areas are colored uniformly by one special grey tone.

In our computations the trajectories are launched from the origin $z = 0$, but the calculations for evaluation of the Lyapunov exponent start after some sufficient number of iterations to exclude the transients.

To begin, let us illustrate the kind of scaling that follows from results of Refs. [10, 11, 5]. In the top part of Fig.2(a) we show a cross-section of the parameter space by surface $u = 0$, $v = 0$, at which the equation (38) reduces to a conventional complex analytic quadratic map (1). What is observed is just a fragment of the Mandelbrot cactus. The GSK critical point is located exactly at the center of the diagram. We select a small box containing the critical point, enlarge and rotate it in accordance with multiplication by $\delta_1 \cong 4.6002 - 8.9812i$, and then reproduce on the bottom plot. In a course of this procedure we redefine the legend for the Lyapunov exponent: as the characteristic time scale triplicates, we decrease by 3 times all the values separating the intervals for coding by definite grey tones. Observe excellent visual

similarity of both pictures.

Next, let us take another cross-section of the parameter space of the map (38), by the surface $(a = 0, b = 0)$. It means that in the parameter space of the original map (28) we are at the manifold $M: C_1(\lambda, \varepsilon) = 0$. The Lyapunov chart is shown in Fig.2(b); the critical point is placed precisely in the middle of the plot. Observe that visually the picture has nothing common with the familiar Mandelbrot set. However, the self-similarity does hold, although it is governed now by the new scaling constant (26). To demonstrate it, we take a box around the critical point. After magnification and rotation corresponding to multiplication by $\delta_2 \cong 2.5872 + 1.8067i$, and with redefinition of the grey-scale coding rule for the Lyapunov exponent, we obtain the chart shown on the bottom plot. It looks remarkably similar to the top diagram.

To give clearer impression of the parameter space arrangement on the manifold M we present in Fig.3 a schema explaining some details of the bifurcation structure. The largest area in the middle corresponds to a stable period-1 state. Here the map possesses the fixed point with two complex conjugate multipliers of modulus less than unity. From the bottom this domain is bounded by a curve, at which the Neimark bifurcation occurs. Here two complex multipliers cross the unit circle. Along the border argument of complex multiplier is varied. For irrational arguments, measured in 2π units, an attractor, which appears as a result of the bifurcation, will be a torus. For rational values it is a resonance cycle (at the sharp ends of the Arnold tongues). At the upper border of the stability domain period-doubling bifurcation takes place. It may be followed either by secondary period doubling (top left part of the border for the period-2 domain), or by Neimark bifurcation (top right part of the border). Note overlap of the period-1 and period-2 areas, which indicates presence of multistability (coexistence of two attractors with distinct basins). At this region the borders of the stability domains appear to be associated with the hard bifurcations (jumps). At the left bottom part of this structure another similarly arranged smaller formation is attached. Here the period-3 regime occurs in the middle part of the stability domain. According to results of the RG analysis and scaling arguments, the sequence of the smaller domains, each of which is attached to its precursor, must be infinite. Asymptotically they become self-similar, accepting a universal form. (In fact, already on the very crude resolution scale of Fig.3 the domains of periods one, three, and nine look remarkably similar.)

The present illustrations support our basic statement that the second eigenvalue δ_2 is of relevance for the non-analytic modification of the map. The scaling properties of the extended parameter space are determined by two constants, the earlier known δ_1 , and the novel δ_2 .

5 Conclusion

In this article we discuss scaling behaviour of complex maps at the accumulation point of period-tripling in the extended parameter space.

We consider the fixed point function of the RG equation of Golberg, Sinai, and Khanin, and analyze a class of perturbations, which are supposed to be small and smooth, but not necessarily obey the Cauchy-Riemann conditions. There is no need in modification of the fixed point equation, but we revise the linearized RG equation, which describes behaviour in a vicinity of the analytic fixed point. Then, we study the eigenvalue spectrum and detect two relevant eigenvectors, one of which is analytic, and another one is non-analytic and violates the Cauchy – Riemann conditions.

To observe the period-tripling universal scaling behaviour in a non-analytic map one needs to make *two* complex coefficients at the relevant eigenvectors equal zero by means of appropriate selection of parameters. Typically, it requires controlling not less than four real parameters. It may be thought that the possibility of observation for such type of behaviour in physical systems is rather problematic. Nevertheless, we can not exclude that the period-tripling criticality may be found in some systems, which possess special symmetries. Apparently, this is an interesting direction of search for physical applications of the complex analytic dynamics.

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Table 1: The coefficients of polynomial expansion for the fixed-point function $g(z)$

1	$1.0 + 0.0i$
z^2	$0.054665304 - 0.749020944i$
z^4	$-0.024397241 - 0.052466461i$
z^6	$-0.002529112 - 0.001197430i$
z^8	$-0.000088081 + 0.000137556i$
z^{10}	$0.000000729 + 0.000018289i$
z^{12}	$0.000000541 + 0.000001194i$
z^{14}	$0.000000074 + 0.000000048i$

Table 2: The coefficients of polynomial expansion for the senior eigenfunction $h^{(1)}(z)$

1	$1.0 + 0.0i$
z^2	$0.181223377 - 0.141361034i$
z^4	$0.002303322 - 0.015962515i$
z^6	$-0.001131785 - 0.000491487i$
z^8	$-0.000151737 + 0.000053131i$
z^{10}	$-0.000010514 + 0.000009686i$
z^{12}	$-0.000000310 + 0.000001085i$
z^{14}	$0.000000033 + 0.000000100i$

Table 3: The coefficients of polynomial expansion for the eigenfunction $h^{(2)}(z, z^*)$ associated with the eigenvalue $\delta_2 = 2.58728651 + 1.80679396i$

	1	z^*	$(z^*)^2$	$(z^*)^4$	$(z^*)^6$	$(z^*)^8$
1	$1.0 + 0.0i$	-3.029846 $+6.351031i$	0.398114 $+0.008514i$	-0.022670 $-0.073276i$	-0.002329 $-0.003512i$	-0.000073 $-0.0002278i$
z^2	1.094504 $-1.043125i$	0.068249 $+1.082134i$	0.086111 $+0.039504i$	-0.001995 $+0.001278i$	-0.000053 $+0.000130i$	0.0000004 $+0.000006i$
z^4	-0.048082 $-0.141290i$	0.053978 $+0.057211i$	0.001900 $-0.001493i$	-0.000124 $+0.000212i$	0.0000003 $+0.000009i$	
z^6	-0.011263 $-0.005046i$	0.004950 $-0.003601i$	-0.000450 $-0.000460i$	-0.000005 $+0.000022i$		
z^8	-0.000616 $+0.000710i$	0.000280 $-0.000811i$	-0.000058 $-0.000028i$	0.0000002 $+0.000002i$		
z^{10}	-0.000002 $+0.000109i$	-0.000004 $-0.000074i$	-0.000004 $+0.0000002i$			

Table 4: Some eigenfunctions of the eigenproblem (15) associated with the infinitesimal variable changes

Variable change, $\Delta \ll 1$	Eigenfunction $h_2(z, z^*)$	Eigenvalue ν
Shift: $z \rightarrow z + \Delta$	$g'(z) - 1$	$\nu = \alpha = -2.0969 + 2.3583i$
Scale change $z \rightarrow z(1 + \Delta)$	$zg'(z) - g(z)$	$\nu = 1$
Non-analytic change $z \rightarrow z + z^* \cdot \Delta$	$z^*g'(z) - (g(z))^*$	$\nu = \alpha/\alpha^* = -0.1169 - 0.9931i$ $ \nu = 1$

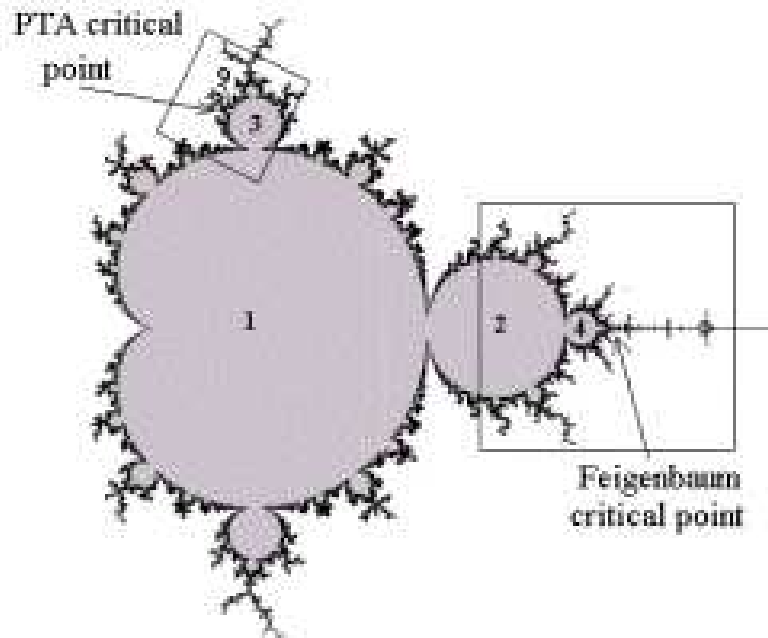


Figure 1: Mandelbrot set for complex analytic quadratic map (1). A unification of grey colored areas is the Mandelbrot cactus. Periods associated with the leaves of the cactus are marked by respective numbers. Two selected boxes show domains of the Feigenbaum period-doubling scaling and of the period-tripling scaling.

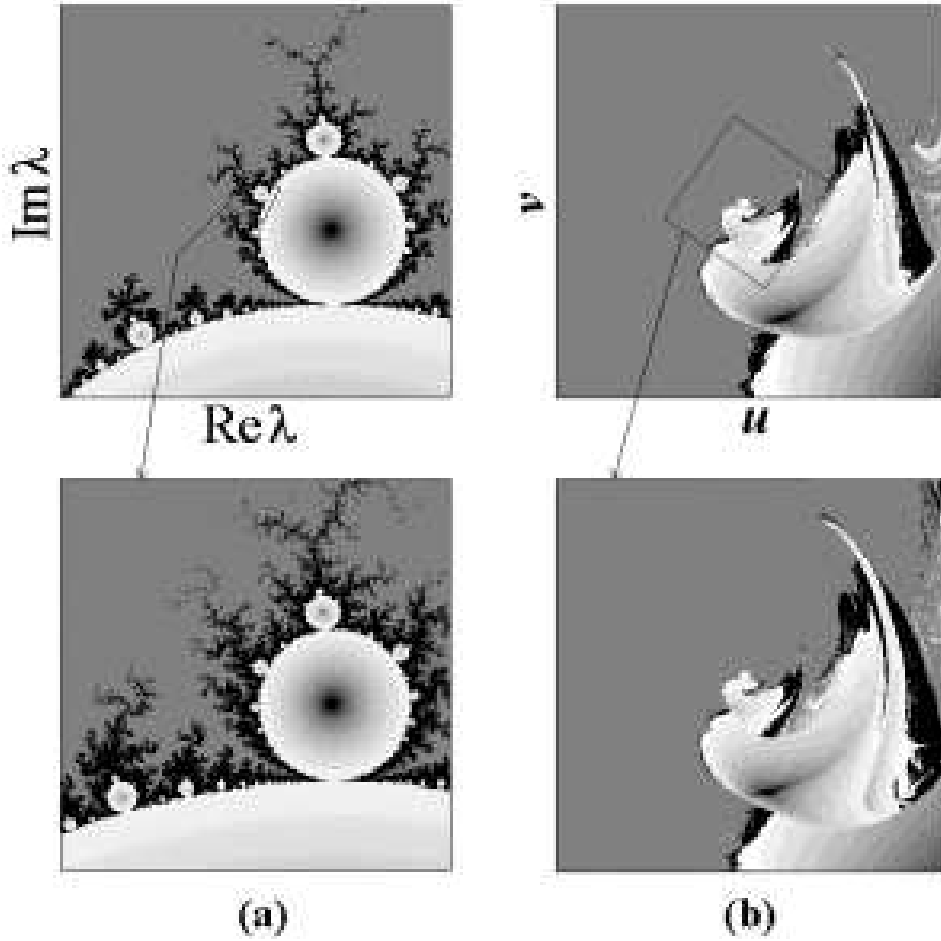


Figure 2: Cross-section of the parameter space by a surface $u = 0, v = 0$, at which the map (38) is complex analytic (a), and by the manifold M , at which the contribution of the senior eigenvector into a perturbation of the RG fixed point is excluded (b). Grey tones from dark to light code values of the Lyapunov exponent from large negative to zero. White color corresponds to zero, and black – to positive Lyapunov exponent. Divergence is shown by uniform coloring with one special grey tone. The GSK critical point is located exactly at the center of the diagrams. A small box containing the critical point, is magnified and rotated in accordance with multiplication by $\delta_1 \simeq 4.6002 - 8.9812i$ on the panel (a) and by $\delta_2 \simeq 2.5873 + 1.8068i$ on the panel (b). The coding rule on the bottom plots is redefined to reveal the self-similarity: all the values separating intervals for coding by definite grey tones are decreased by 3 times.

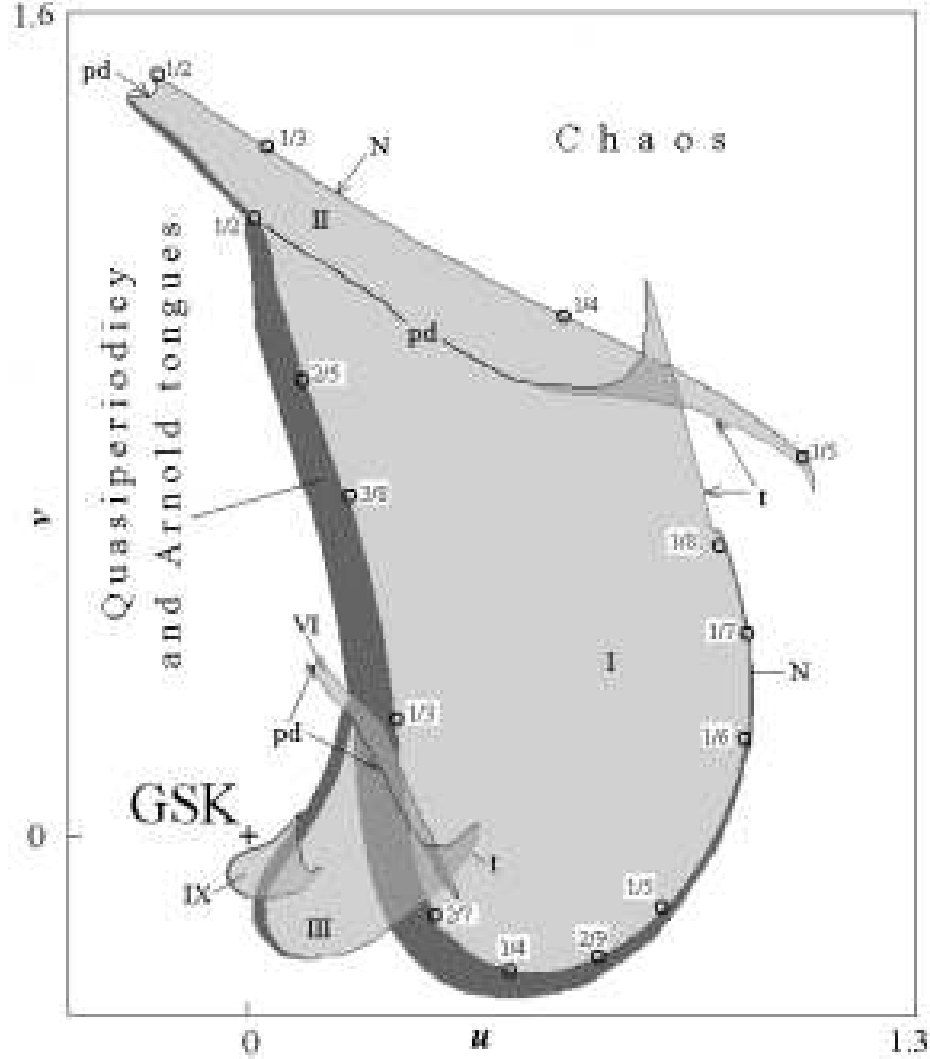


Figure 3: A schema of the parameter space arrangement in the cross-section of the extended parameter space by the manifold M , at which the contribution of the senior eigenvector into a perturbation of the RG fixed point is excluded. Bifurcation lines are marked as pd (period-doubling), t (tangent), N (Neimark). Light grey areas correspond to periodic regimes of periods marked by the Roman figures. Dark grey designates the area of quasi-periodicity and Arnold tongues (not shown). On the Neimark bifurcation curve some points of rational arguments of the multiplier are marked, these are the sharp ends of the respective Arnold tongues. The GSK critical point is marked by a cross.